nitrogen previously liberated in the ratio of 3 to 2. One-third of the nitrogen contained in the dichloro urea used is found as ammonia.

The investigation of dichloro urea, which is an extremely reactive body and promises to be of considerable use in organic synthesis, is being continued.

The thanks of the author are due to Dr. Baker for allowing him to use the Christ Church Laboratory, where this work has been carried out.

Note on the Instability of Tubes subjected to End Pressure, and on the Folds in a Flexible Material.

By A. Mallock, F.R.S.

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When a straight rod is subjected to end compression it is stable for small lateral displacements unless the compressing force exceeds a definite limit depending on the elastic constants of the material of the rod and its length and cross section dimensions.

If this limit is exceeded, the rod is unstable and the least departure from straightness grows under the action of the force, the axis of the rod then taking the form of one of the well-known elastic curves; and this is the only form which a solid rod can take in the circumstances.

With tubes and plates, however, the case is different, for with the tube the ratio of the thickness of the walls to the diameter of the tube has to be considered as well as the ratio of the diameter to the length. Thus a tube whose length is insufficient to produce instability involving a bending of the axis may become unstable by the crumpling up of the walls, the axis itself remaining straight.

In plates something of the same kind may take place. The edges of a rectangular plate may be constrained to remain straight and parallel to one another, but if pressure is applied to two opposite edges instability will ensue if it exceeds a critical value.

In the case of solid rods the governing condition is the constancy (to the first order) of the length of the axis; with tubes and plates it is the constancy to the same order of the area of the mid-wall surface. Considering the case of tubes in rather more detail, take the axis of the tube as z and let its unstrained radius be r_0.

Under end compression the surface may become unstable by deformation into any of the cylindrical harmonics of the type

$$r = r_0 + a \cos n\,\theta \cos \frac{2\pi}{\lambda} z,$$

where θ is the angle which r makes with a fixed diameter of the tube and λ the length of the fold parallel to the axis. The order of the harmonic which will naturally be assumed by the deformed tube depends on the ratio (h/r) of the thickness of the walls to the diameter and will be such that the potential energy of the combined bending and shearing involved may be a maximum.

I will not in this note work out individual cases, but it will be seen that the smaller the ratio h/r the higher will be the order of the harmonic, because, since the shear becomes relatively more important as h/r diminishes, λ/r must also diminish to fulfil the condition of maximum potential energy.*

If the crushing is continued until the tube is greatly reduced in length the folds are seen to develop into the symmetrical shapes shown in the photographs (figs. 1, 2, 3), for which $n = 1$, 2, and 3 respectively. For $n = 1$ the folds

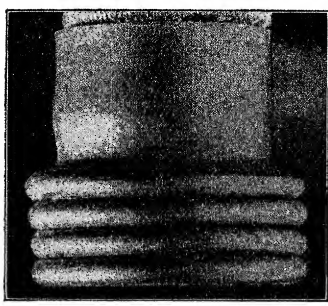


Fig. 1.

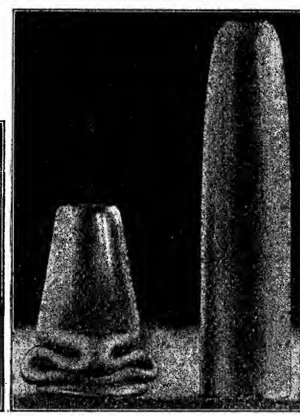


Fig. 2.

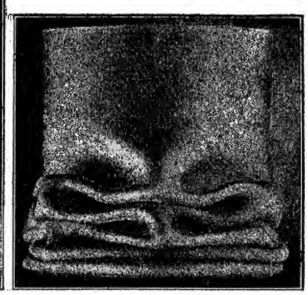


Fig. 3.

are circular in plan and independent of θ; when $n = 2$ the plan of the folds is a square, and when $n = 3$ the plan is hexagonal.

It may be noticed that the instability always shows itself first at one end, and that since the reaction against end pressure decreases as the deformation goes on, each fold is completed in succession, the next not becoming marked until the reaction is increased by the previous fold resting against the last but one.

* It often happens that owing to the constraint applied by the surfaces between which the tube is crushed, the fold first formed is of the first order, even when the ratio of thickness to diameter is such that a higher order is the natural one.

The crushing force requisite therefore undergoes periodic variations, being a maximum at the beginning of the formation of a new fold and a minimum when the fold is nearly completed.

If we examine the completed folds on the assumption that the extension of any element of the surface is small compared to the depth of the fold, it will be seen that the side of each fold (AB), see fig. 4 (for which $n = 3$), at the

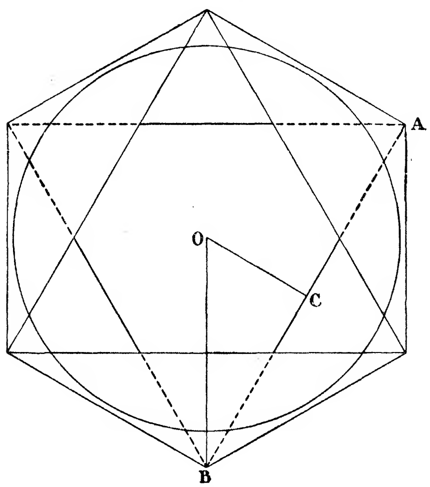


FIG. 4.

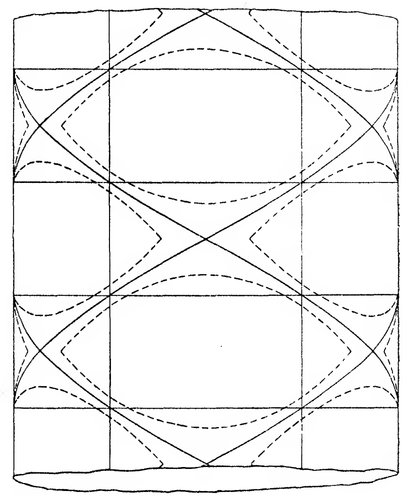


FIG. 5.

re-entrant angle has a length $2\pi r/n$. Also that the least distance (OC) of the fold from the centre is $\frac{\pi r}{n}\cos\frac{\pi}{n}$ and the maximum distance (OB) is $\frac{\pi r}{n}\sin\frac{\pi}{n}$. The length of each fold as it would appear on the undeformed tube is therefore $2\pi r\,(1-\cos\frac{\pi}{n})n\sin\frac{\pi}{n}$. Thus for $n = 2$, $\lambda = \pi r$; and for $n = 3$, $\lambda = 2\pi r/3\sqrt{3}$. The formulæ do not apply when $n = 1$, for in this case the assumption that the extension is small compared to the depth of the fold is untenable. But with $n = 2$ and $n = 3$ I find by trial that the number of folds formed from a given length of tube corresponds very fairly with the supposition that the extension is small.

In fig. 5 the vertical and horizontal lines are the nodal lines of the harmonic $\cos 3\theta \cos 3\sqrt{3}\,z$ on a cylindrical surface; the spiral lines ultimately become the salient angles of the folds, and the dotted lines indicate the locus of points of no compression or extension when the tube is completely crushed.

The potential due to small deformations of either the tubular or the completely crushed surface can be calculated by the use of recognised

functions, but the intermediate stages, where the local extension or compression of the mid-wall surface is not a small quantity, present a far more difficult problem, the solution of which would probably require the investigation of functions of a new class.

The problem is, in fact, merely a particular case of the general theory of the form of the folds of drapery, that is of the surfaces into which a plane can be deformed if its material can be stretched by an amount which is finite but small compared to the depth of the folds.

As a simple example, consider the case of a circular cone (fig. 6). Let this

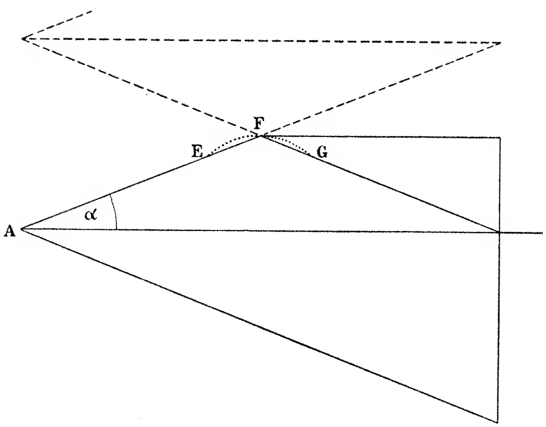


Fig. 6.

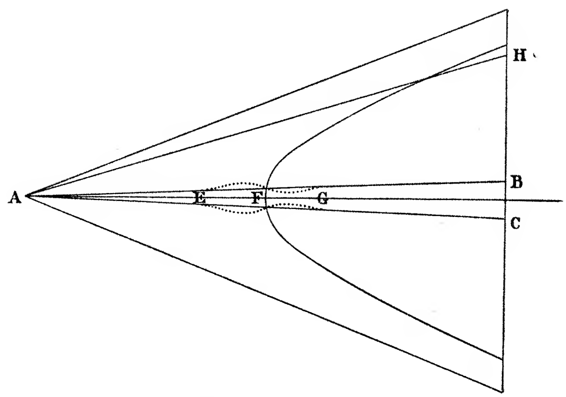


Fig. 7.

be intersected by a similar cone, the axes of both being in the same plane, and intersecting at equal distances from their apices. The curve of the intersection of the surfaces is a conic whose plane bisects the angle between the axes. Now let that part of one of the cones outside the curve of intersection be removed; the remaining surface is of exactly the same area as the original cone and could have been formed from it by bending without any stretching of the mid-surface. Such a partly inverted cone is typical of the fold in a flexible and inextensible surface, and it remains to be seen what form the angle of the fold would take if the material resisted bending. In this case there cannot be an abrupt change in the direction of the surface in passing from the original cone to the part inverted, and what in the perfectly thin and flexible material was a sharp crease becomes, in virtue of the stiffness, a rounded curve, EFG.

If AB, AC, fig. 7, are two of the generating lines of the cone, a small distance on either side of that passing through the vertex of the conic, the effect of resistance to bending will be to increase the distance of the surface from the axis of the original cone from E to F and to diminish the distance

of the surface from the axis of the inverted cone from F to G. Hence there will be circumferential extension of the material from E to F and circumferential compression from F to G.

The distorted generating lines will therefore take some such shape as is indicated by the dotted lines in fig. 7.

In the undistorted cones the principal radii of curvature at the vertex of the conic are, for the original cone, $-\infty$ and AF $\tan\alpha$ and for the inverted surface $+\infty$ and $-$AF $\tan\alpha$. By a general theorem relating to the curvature of surfaces (the measure of curvature being $(R_1R_2)^{-1}$, where R_1R_2 is the product of the principal radii of the surface), no stretching of the surface is involved by any changes in the principal radii of curvature which satisfy the condition $R_1R_2 =$ constant. In the neighbourhood of the vertex of the conic the resistance to bending makes both the principal radii of curvature finite, hence $(R_1R_2)^{-1}$ is finite (instead of zero as on the undistorted surfaces), and if s be any small area on the distorted surface, the amount of stretching due to the resistance to bending is $(1-\cos\theta)/(1+\cos\theta)$,* where θ is the average angle which the edge of the distorted area makes with the original surface.

Where, as at AH, the generating lines cut the conic very obliquely, only one of the principal radii of curvature is appreciably affected, and the resistance to bending has hardly any effect in altering the area of the surface.

Thus absence of perfect flexibility causes a general rounding off of the sharp crease which forms the conic on the undistorted surface, a rounding off which is more and more marked as the distance from the vertex increases, and in addition to this, a knuckle-like prominence is produced in the neighbourhood of the vertex itself.

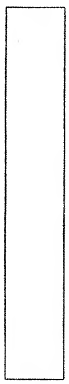
As a second example of simple folds, it may be seen at once that any rectangular plane surface can be folded without stretching into a series of cones such as are shown in photographs Nos. 8 and 9. In No. 9 the radius of curvature has the same sign in all the cones, but in No. 8 the curvature is alternately positive and negative. In both, when seen in plan, the average direction of the edges at right angles to the axes of the cones is parallel to direction of the corresponding edges of the plane before folding. When viewed in elevation, however (figs. 8A and 9A), it will be noticed that the free edges of the folds appear to be at right angles to the original plane in No. 8, where the curvature is alternately positive and negative, but at right angles to the slant side of the cones in No. 9.†

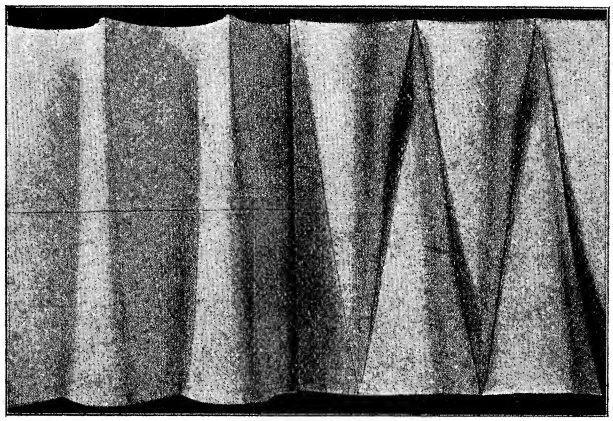
* The stretching is not uniform over the area, but increases from 0 to a maximum at the boundary.

† It should be stated that the edges do not really lie in a plane, but the departure from a plane is of a very small order.

If, then, any number of equal rectangular sheets are folded as in fig. 8, they may be joined at the free edges of the folds without stretching and the average surface of the joined sheets will be a plane.

Sheets folded as in No. 9 may also have the free edges of their folds joined, but each sheet will now make an angle with the adjacent one equal to twice the angle which the slant side of the cones makes with the plane from which they were formed. Sheets thus joined are shown in the photograph (fig. 10) and in plan in fig. 10A.





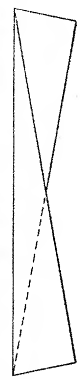


FIG. 8A. FIG. 8. FIG. 9. FIG. 9A

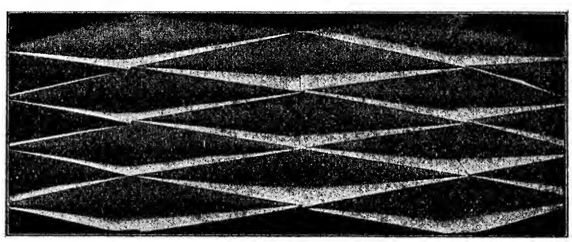


FIG. 10.

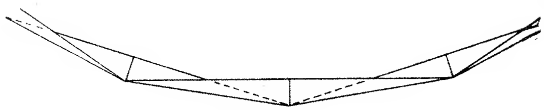


FIG. 10A.

Thus the cones of the combination of folded planes will all touch a cylinder, and the character of the surface of the combination will be analogous to, but not identical with, the intermediate stages of crushing of a tube by end pressure.

Fig. 1.

Fig. 2.

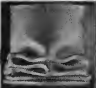


FIG. 3.

Fig. 8a.

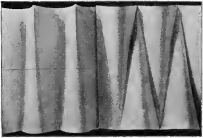


Fig. 8.

Fig. 9.

Fig. 9a

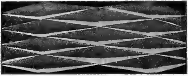


Fig. 10.

Fig. 10a.